

TOTALLY UMBILICAL HYPERSURFACES OF MANIFOLDS ADMITTING A UNIT KILLING FIELD

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ABSTRACT. We prove that a Riemannian product of type $\mathbb{M}^n \times \mathbb{R}$ admits totally umbilical hypersurfaces if and only if \mathbb{M}^n has locally the structure of a warped product and we give a complete description of the totally umbilical hypersurfaces in this case. Moreover, we give a necessary and sufficient condition under which a Riemannian three-manifold carrying a unit Killing field admits totally geodesic surfaces and we study local and global properties of three-manifolds satisfying this condition.

1. INTRODUCTION

The starting point of the research leading to this paper was the classification of totally umbilical surfaces in three-dimensional homogeneous spaces with a four-dimensional isometry group, which can be found in [11] and [12]. As is well-known, these three-spaces admit Riemannian submersions onto surfaces of constant Gaussian curvature and the unit vector field tangent to the fibers is Killing. It turns out that such a space admits totally umbilical surfaces if and only if it is a Riemannian product of the base surface and the fibers, i.e., if and only if its universal covering is either $\mathbb{S}^2(\kappa) \times \mathbb{R}$ or $\mathbb{H}^2(\kappa) \times \mathbb{R}$. Moreover, the obtained classification of totally umbilical surfaces was extended to a classification of totally umbilical hypersurfaces of the conformally flat symmetric manifolds $\mathbb{S}^n(\kappa) \times \mathbb{R}$ and $\mathbb{H}^n(\kappa) \times \mathbb{R}$ in [13] and [3].

Two questions for further generalizations come up naturally now.

- (1) When does a Riemannian product of type $\mathbb{M}^n \times \mathbb{R}$ admit totally umbilical hypersurfaces and what are they?
- (2) When does a Riemannian three-space with a unit Killing field admit totally umbilical surfaces and what are they?

In this paper, we give a complete answer to the first question. Our Theorem 1 states that a necessary and sufficient condition for $\mathbb{M}^n \times \mathbb{R}$ to admit totally umbilical hypersurfaces is that \mathbb{M}^n itself has (locally) the structure of a warped product. Moreover, we give a full description of all totally umbilical hypersurfaces of such a manifold and we remark that our results are still valid if we start with a warped product instead of with a Riemannian product as ambient space.

For the second question we give a partial answer. We find a necessary and sufficient condition for a three-manifold with a unit Killing field to admit totally

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geodesic surfaces. Remark that it is not necessary that the three-manifold reduces to a Riemannian product, a fact which is already illustrated by the standard three-sphere. We describe the totally geodesic surfaces and we study the local and global properties of three-spaces satisfying our condition.

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2. PRELIMINARIES

Let $(M^n, g) \rightarrow (\tilde{M}^{n+1}, \tilde{g})$ be an isometric immersion between Riemannian manifolds. If N is a unit normal vector field along the immersion and ∇ and $\tilde{\nabla}$ are the Levi-Civita connections of (M^n, g) and $(\tilde{M}^{n+1}, \tilde{g})$, then the second fundamental form h and the shape operator S associated to N are defined by the formulas of Gauss and Weingarten: for any vector fields X, Y on M^n one has

$$\tilde{\nabla}_X Y = \nabla_X Y + h(X, Y)N, \quad SX = -\tilde{\nabla}_X N.$$

It is easy to check that S is a symmetric $(1, 1)$ -tensor field on M^n , which is related to h by $h(X, Y) = g(SX, Y)$. We call the immersion *totally umbilical* if S is a multiple of the identity at every point and we call it *totally geodesic* if S vanishes identically.

In our results, some special types of vector fields on Riemannian manifolds will occur. Let (M, g) be a Riemannian manifold and let ξ be a vector field on M . Then ξ is said to be *Killing* if and only if $\mathcal{L}_\xi g = 0$, where \mathcal{L} is the Lie derivative. This condition means that the flow of ξ consists of isometries, and in terms of the Levi-Civita connection ∇ one can reformulate it as

$$g(\nabla_X \xi, Y) + g(\nabla_Y \xi, X) = 0$$

for all $p \in M$ and $X, Y \in T_p M$.

More generally, ξ is said to be *conformal* if and only if $\mathcal{L}_\xi g = 2\phi g$ for some function ϕ . This means that the flow of ξ consists of conformal maps.

Finally, we say that ξ is *closed conformal* if and only if it is conformal and its dual one-form is closed. It can be checked by a straightforward computation that ξ is closed conformal if and only if

$$\nabla_X \xi = \phi X,$$

for all $p \in M$ and all $X \in T_p M$, where ϕ is as above.

In all what follows the manifolds will be assumed of class C^∞ .

3. TOTALLY UMBILICAL HYPERSURFACES OF MANIFOLDS OF TYPE $\mathbb{M}^n \times I$

Denote by $\mathbb{M}^n \times I$ the Riemannian product of a Riemannian manifold $(\mathbb{M}^n, g_{\mathbb{M}^n})$ and an open interval I of the Euclidean line and let $\pi : \mathbb{M}^n \times I \rightarrow \mathbb{M}^n$ be the canonical projection. We shall denote by ξ a unit vector field on $\mathbb{M}^n \times I$, tangent to the fibres of π . Remark that ξ is a unit Killing field.

There are two natural families of examples of totally geodesic hypersurfaces of $\mathbb{M}^n \times I$, namely the slices $\mathbb{M}^n \times \{t_0\}$, $t_0 \in I$ and the inverse images under π of totally geodesic hypersurfaces of \mathbb{M}^n , if they exist. We are thus interested in totally umbilical hypersurfaces which are at some point neither orthogonal nor tangent to ξ .

If Σ is a hypersurface of $\mathbb{M}^n \times I$ with unit normal N , one can define a vector field T and a real-valued function ν on Σ by the following orthogonal decomposition of ξ :

$$(1) \quad \xi = T + \nu N.$$

Then T and ν satisfy the following equations.

Lemma 1. *Let Σ be a hypersurface in $\mathbb{M}^n \times I$ and denote by ∇ the Levi-Civita connection of Σ , by S the shape operator of the immersion and by h the second fundamental form. Then for any vector X tangent to Σ :*

$$\nabla_X T = \nu S X, \quad X(\nu) = -h(X, T).$$

Proof. Denote by $\tilde{\nabla}$ the Levi-Civita connection of $\mathbb{M}^n \times I$. Since ξ is a parallel vector field on $\mathbb{M}^n \times I$, one has $\tilde{\nabla}_X \xi = 0$. Now let X be tangent to Σ . By the definitions of T and ν and by using the formulas of Gauss and Weingarten, it follows that

$$0 = \tilde{\nabla}_X \xi = \tilde{\nabla}_X (T + \nu N) = \nabla_X T + h(X, T)N + X(\nu)N - \nu S X.$$

The result follows by considering the tangent, resp. normal, component of the above equation. \square

If Σ is totally umbilical in $\mathbb{M}^n \times I$, say $S = \lambda \text{id}$, then T is a closed conformal field on Σ . Indeed, in this case it follows from Lemma 1 above that $\nabla_X T = \nu \lambda X$ for all vector fields X tangent to Σ . We shall now prove that if Σ is non-vertical and non-horizontal at some point p , we can use T to construct a local non-vanishing conformal field on \mathbb{M}^n .

Proposition 1. *Let Σ be a totally umbilical hypersurface of $\mathbb{M}^n \times I$, which is neither vertical nor horizontal at some point. Then the canonical projection $\pi : \mathbb{M}^n \times I \rightarrow \mathbb{M}^n$ is locally a diffeomorphism between an open neighborhood U of this point in Σ and the open subset πU of \mathbb{M}^n . Let T be as above and denote by T_0 be the projection of T to πU , rescaled such that it has the same length as T again. Then T_0 is a closed conformal field on πU .*

Proof. Let Σ be a totally umbilical hypersurface of $\mathbb{M}^n \times I$. Let ξ , N , T and ν be as above and assume that the shape operator associated to N is $S = \lambda \text{id}$. Suppose Σ is non-vertical and non-horizontal at some point, then it is clear that there is an open neighborhood U of this point in Σ where ν does not vanish and such that π is a local diffeomorphism between U and its image πU in \mathbb{M}^n . First, extend the vector fields T , N and the functions ν , λ to the whole of $\pi U \times I$ by using the one-parameter group of translations corresponding to the Killing field ξ and denote these again by T , N , ν and λ . Since ν and λ are constant on fibres of π , one can also view them as functions on πU . Using these notations, the vector field T_0 on πU is

$$T_0 = (d\pi)(T) \frac{\|T\|}{\|(d\pi)(T)\|} = \frac{1}{\nu} (d\pi)(T)$$

and its horizontal lift to $\pi U \times I$ is

$$\widetilde{T_0} = \nu \xi - N = \frac{\nu^2 - 1}{\nu} \xi + \frac{1}{\nu} T.$$

Remark that T_0 is, up to the sign, the projection of N to πU .

Now let X be a vector field on πU and denote by \tilde{X} its horizontal lift. Then

$$(2) \quad \nabla_X^{\mathbb{M}^n} T_0 = (d\pi)(\tilde{\nabla}_{\tilde{X}} \widetilde{T_0}) = (d\pi)(\tilde{\nabla}_{\tilde{X}} (\nu \xi - N)) = -(d\pi)(\tilde{\nabla}_{\tilde{X}} N)$$

Let Y be a local vector field on Σ such that $(d\pi)(Y) = X$. Denote the extension to $\pi U \times I$, using the flow of ξ , again by Y . Then $\tilde{X} = Y - \langle Y, \xi \rangle \xi$ and

$$(3) \quad \tilde{\nabla}_{\tilde{X}} N = \tilde{\nabla}_{Y - \langle Y, \xi \rangle \xi} N = \tilde{\nabla}_Y N - \langle Y, \xi \rangle \tilde{\nabla}_\xi N = -\lambda Y.$$

Here we used that $[\xi, N] = 0$ implies $\tilde{\nabla}_\xi N = \tilde{\nabla}_N \xi = 0$. From (2) and (3), we obtain $\nabla_X^{\mathbb{M}^n} T_0 = \lambda X$, which proves that T_0 is indeed closed conformal. \square

The fact that \mathbb{M}^n admits a local closed conformal field, determines locally its Riemannian structure, as shown by the following known result (see [6, 9] and the references therein).

Proposition 2. *Let V be a local closed conformal field without zeros on a Riemannian manifold \mathbb{M}^n , say $\nabla_X^{\mathbb{M}^n} V = fX$ for some non-vanishing function f and for all vector fields X on \mathbb{M}^n . Then \mathbb{M}^n has locally the structure of a warped product of an interval of the Euclidean line with some $(n-1)$ -dimensional Riemannian manifold.*

Proof. One can check that the distribution orthogonal to V is integrable and hence one can find a local coordinate system (x_1, \dots, x_n) on \mathbb{M}^n such that $\partial_{x_1} = V$ and ∂_{x_j} is orthogonal to ∂_{x_1} for $j \geq 2$. With respect to these coordinates, the metric on \mathbb{M}^n takes the form

$$g = g_{11}(x_1, \dots, x_n) dx_1^2 + \sum_{i,j=2}^n g_{ij}(x_1, \dots, x_n) dx_i dx_j.$$

It follows from a straightforward computation that $\partial_{x_j} g_{11} = 0$ for $j \geq 2$ and that $\partial_{x_1} g_{ij} = 2f g_{ij}$ for $i, j \geq 2$. Hence, one has

$$g = g_{11}(x_1) dx_1^2 + \exp\left(2 \int f dx_1\right) \sum_{i,j=2}^n c_{ij}(x_2, \dots, x_n) dx_i dx_j.$$

To conclude, we prove that $\partial_{x_j} f = 0$ for $j \geq 2$, such that, after a change of the x_1 -coordinate, the metric above is indeed a warped product metric. To see this, let R be the curvature tensor of \mathbb{M}^n , then

$$\begin{aligned} 0 &= \langle R(\partial_{x_1}, \partial_{x_j}) \partial_{x_1}, \partial_{x_1} \rangle = \langle \nabla_{\partial_{x_1}}^{\mathbb{M}^n} \nabla_{\partial_{x_j}}^{\mathbb{M}^n} \partial_{x_1} - \nabla_{\partial_{x_j}}^{\mathbb{M}^n} \nabla_{\partial_{x_1}}^{\mathbb{M}^n} \partial_{x_1}, \partial_{x_1} \rangle \\ &= \langle (\partial_{x_1} f) \partial_{x_j} - (\partial_{x_j} f) \partial_{x_1}, \partial_{x_1} \rangle = -(\partial_{x_j} f) g_{11}. \end{aligned}$$

\square

Remark 1. The converse to Proposition 2 is also true. In a warped product $I \times_f M = (I \times M, dt^2 + f(t)^2 g_M)$, the field $f(t) \partial_t$ is closed conformal and vanishes nowhere.

We can now prove our main result in this section.

Theorem 1. *A Riemannian product space $\mathbb{M}^n \times I$ admits a totally umbilical hypersurface Σ , which is neither vertical nor horizontal at some point $(p, t) \in \mathbb{M}^n \times I$, if and only if \mathbb{M}^n has in a neighborhood of p the structure of a warped product of an interval of the Euclidean line with some $(n-1)$ -dimensional Riemannian manifold.*

In particular, when $n = 2$, there exists a totally umbilical surface in $\mathbb{M}^2 \times I$, which is neither vertical nor horizontal at some point $(p, t) \in \mathbb{M}^2 \times I$, if and only if \mathbb{M}^2 admits a non zero Killing field in a neighborhood of p . Moreover any such surface is invariant by a local one-parameter group of local isometries of $\mathbb{M}^2 \times I$ keeping the factor I pointwise fixed.

Proof. It follows from Propositions 1 and 2 above that \mathbb{M}^n having the structure of a warped product in a neighborhood of p is a necessary condition for $\mathbb{M}^n \times I$ to admit a totally umbilical hypersurface which is non-vertical and non-horizontal at (p, t) .

We shall now prove that this condition is also sufficient. Assume that $\mathbb{M}^n = J \times_f M^{n-1}$, i.e., that the metric on \mathbb{M}^n can be written as

$$g_{\mathbb{M}^n} = dx_1^2 + f(x_1)^2 g_{M^{n-1}}(x_2, \dots, x_n).$$

Then the metric on $\mathbb{M}^n \times I$ can be written as

$$g = dx_0^2 + dx_1^2 + f(x_1)^2 g_{M^{n-1}}(x_2, \dots, x_n).$$

We know from above that a non-vertical and non-horizontal totally umbilical hypersurface Σ of $\mathbb{M}^n \times I$ should be tangent to the distribution orthogonal to the vector fields ∂_{x_0} and ∂_{x_1} at any of its points. This means that Σ is generated by a curve in the (x_0, x_1) -plane, say $\alpha(s) = (x_0(s), x_1(s))$. Assume that α is parametrized by arc length, then there exists a function θ such that

$$x'_0(s) = \sin \theta(s), \quad x'_1(s) = \cos \theta(s).$$

In this case, the tangent space to Σ is spanned by $X_1 = \sin \theta(s) \partial_{x_0} + \cos \theta(s) \partial_{x_1}$, $X_2 = \partial_{x_2}, \dots, X_n = \partial_{x_n}$ and a unit normal to Σ is given by $N = \cos \theta(s) \partial_{x_0} - \sin \theta(s) \partial_{x_1}$.

One can compute the Levi-Civita connection $\tilde{\nabla}$ of $\mathbb{M}^n \times I$ from the metric above to verify

$$\tilde{\nabla}_{X_1} N = -\theta'(s) X_1, \quad \tilde{\nabla}_{X_j} N = -\sin \theta(s) \frac{f'}{f} X_j$$

for every $j \geq 2$. Hence, Σ is totally umbilical if and only if

$$\theta'(s) = \sin \theta(s) \frac{f'}{f}.$$

One can now use this equation to determine the functions $x_0(s)$ and $x_1(s)$. Indeed, we have

$$x''_1(s) = -\sin \theta(s) \theta'(s) = -\sin^2 \theta(s) \frac{f'(x_1(s))}{f(x_1(s))} = -(1 - x'_1(s)^2) \frac{f'(x_1(s))}{f(x_1(s))},$$

which yields after a first integration

$$x'_1(s) = \pm \sqrt{1 - c^2 f(x_1(s))^2}$$

for some real constant c . This ODE for $x_1(s)$ is, at least locally, always solvable. The function $x_0(s)$ is then determined by

$$x_0(s) = \int \sqrt{1 - x'_1(s)^2} ds = \int c f(x_1(s)) ds.$$

Thus there does always exist a non-vertical and non-horizontal totally umbilical hypersurface of $\mathbb{M}^n \times I$ if \mathbb{M}^n is locally isometric to the warped product described above. A parametrization for such a totally umbilical hypersurface Σ is $\varphi(s, u_1, \dots, u_{n-1}) = (x_0(s), x_1(s), u_1, \dots, u_{n-1})$.

In the particular case when $n = 2$, observe the following general fact that can be checked straightforwardly. Let J denote the rotation over 90 degrees of an oriented Riemannian surface M^2 , which is locally well-defined on any Riemannian surface M^2 . Then a vector field X on M^2 is closed conformal if and only if JX is Killing. Hence, if Σ is a totally umbilical surface in $\mathbb{M}^2 \times I$, then JT is a Killing field on

Σ . Moreover, JT is orthogonal to the fibers of π and $(d\pi)(JT) = JT_0$ is a Killing field on \mathbb{M}^2 . This implies the result. \square

In particular, from Theorem 1, we recover the classification of totally umbilic surfaces in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$ obtained in [11] and [12].

Remark 2. Theorem 1 also gives a classification of totally umbilical hypersurfaces in Riemannian product spaces of type $\mathbb{M}^n \times I$.

Remark 3. As a further particular case, it is interesting to observe that the results above provide a new proof for the classification of totally umbilic hypersurfaces in the Euclidean space \mathbb{R}^{n+1} since the latter can be viewed (in various ways) as a product $\mathbb{R}^n \times \mathbb{R}$. The present proof has the advantage to work assuming only a C^2 regularity for the hypersurfaces. It can indeed easily be checked that C^2 regularity for the hypersurfaces is enough in the above results. The standard proof of the classification of totally umbilic hypersurfaces in \mathbb{R}^{n+1} works for hypersurfaces with at least C^3 regularity. It is known this classification also holds for C^2 -hypersurfaces (see [7] and [10]).

Corollary 1. *A Riemannian warped product space $I \times_f \mathbb{M}^n$ admits a totally umbilical hypersurface Σ , which is neither vertical nor horizontal at some point $(t, p) \in I \times_f \mathbb{M}^n$, if and only if \mathbb{M}^n has in a neighborhood of p the structure of a warped product of an interval of the Euclidean line with some $(n-1)$ -dimensional Riemannian manifold.*

Proof. First observe that a Riemannian warped product space $I \times_f \mathbb{M}^n$ is conformal to a Riemannian product $J \times \mathbb{M}^n$, for some open interval J of \mathbb{R} . Indeed, with some abuse of notation, the metric on $I \times_f \mathbb{M}^n$ writes

$$g = dt^2 + f(t)^2 g_{\mathbb{M}^n} = f(t)^2 \left(\frac{dt^2}{f(t)^2} + g_{\mathbb{M}^n} \right).$$

Choosing a new parameter $s = \psi(t) = \int \frac{dt}{f(t)}$ and introducing the function $h(s) = f(\psi^{-1}(s))$, we can write:

$$g = h(s)^2 (ds^2 + g_{\mathbb{M}^n})$$

as desired.

Now the claim follows from Theorem 1 and the known fact that totally umbilic hypersurfaces are preserved under conformal diffeomorphisms between the ambient manifolds. \square

4. TOTALLY GEODESIC SURFACES IN A THREE-DIMENSIONAL SPACE ADMITTING A UNIT KILLING FIELD

In this section we will characterize locally the Riemannian three-manifolds admitting a unit Killing field which possess totally geodesic surfaces. A good reference on manifolds admitting a Killing field of constant length is given by [1].

We start with a result which is valid in all dimensions. Let M denote a Riemannian manifold which admits a unit Killing field ξ . The product manifolds $\mathbb{M}^n \times I$ considered in Section 3 are a particular case. Denote by $\tilde{\nabla}$ the Levi-Civita connection of M . Let Σ be a hypersurface in M with unit normal N . Then we can, as in

the previous section, define a vector field T and a real-valued function ν on Σ by the orthogonal decomposition

$$(4) \quad \xi = T + \nu N.$$

The following result is a key fact for our purposes:

Proposition 3. *Let Σ be a totally geodesic hypersurface in a Riemannian manifold M admitting a unit Killing field ξ . Suppose ξ is not tangent to Σ at some point. Then one can extend the vector field T to a neighborhood of this point in M using the local flow of ξ . If one denotes the resulting vector field again by T , then T is a local Killing field on M .*

Proof. Since ξ is transversal to Σ in a neighborhood of the point, using the local flow $(\phi_t)_{t \in I}$ of ξ we obtain a foliation \mathcal{F} of an open subset of M by the totally geodesic hypersurfaces $\phi_t(\Sigma)$. In this way we have local extensions of the fields T and N and the function ν . Denote these extensions again by T , N and ν . Note that (4) is again valid for these extensions.

We have to verify that $\langle \tilde{\nabla}_X T, Y \rangle + \langle X, \tilde{\nabla}_Y T \rangle = 0$ for all vector fields X and Y .

First we note that

$$(5) \quad T(\nu) = 0.$$

Indeed, since the hypersurfaces $\phi_t(\Sigma)$ are totally geodesic and ξ is a unit Killing field, we have

$$T(\nu) = T(\langle \xi, N \rangle) = \langle \tilde{\nabla}_T \xi, N \rangle = \langle \tilde{\nabla}_T \xi, \frac{1}{\nu}(T - \xi) \rangle = 0.$$

Consider now a vector field X which is tangent to the leaves of the foliation \mathcal{F} . Then

$$(6) \quad \tilde{\nabla}_X T = \tilde{\nabla}_X \xi - X(\nu)N.$$

Furthermore,

$$\tilde{\nabla}_N T = \tilde{\nabla}_{\frac{1}{\nu}(\xi - T)} T = \frac{1}{\nu}(\tilde{\nabla}_T \xi - \tilde{\nabla}_T T).$$

Using (6) and (5) we get

$$(7) \quad \tilde{\nabla}_N T = 0.$$

Using (6) and (7), it is easy to check that T is Killing. \square

We now particularize to the case where the ambient manifold is three-dimensional. Well-known examples of such manifolds are Riemannian products of type $\mathbb{M}^2 \times \mathbb{R}$ and also the unit three-sphere \mathbb{S}^3 , Berger spheres and the Thurston spaces $\widehat{\text{SL}}(2, \mathbb{R})$ and Nil_3 . In the next section we will describe more such spaces. We first prove an important basic formula.

Lemma 2. *Let M^3 be an oriented Riemannian manifold carrying a unit Killing field ξ . Denote by $\tilde{\nabla}$ the Levi-Civita connection of M^3 and by \times its cross product. Then there exists a real-valued function τ on M^3 , with $\xi(\tau) = 0$, such that*

$$\tilde{\nabla}_X \xi = \tau(X \times \xi)$$

for all vector fields X on M^3 .

Proof. It is clear that $\tilde{\nabla}_X \xi$ is perpendicular to ξ and X since ξ is a unit Killing field. Because the space is three-dimensional, we obtain that $\tilde{\nabla}_X \xi = \tau(X)(X \times \xi)$ for some real number $\tau(X)$.

Since the mapping $T_p M^3 \rightarrow T_p M^3 : X \mapsto \tilde{\nabla}_X \xi = \tau(X)(X \times \xi)$ must be linear for every point p , it is easily seen that τ can only depend on the choice of $p \in M^3$ and not on the choice of $X \in T_p M^3$. Hence τ is a real-valued function on M^3 .

To see that this function satisfies $\xi(\tau) = 0$, let $(\phi_t)_{t \in I}$ be the local flow of ξ as above. Then $\tilde{\nabla}_{(d\phi_t)X}(d\phi_t)\xi = (d\phi_t)(\tilde{\nabla}_X \xi)$, or, equivalently, $\tau(\phi_t(p))((d\phi_t)X \times \xi) = \tau(p)((d\phi_t)X \times \xi)$ for every parameter t and for every $p \in M^3$ and $X \in T_p M^3$. We conclude that $\tau(\phi_t(p)) = \tau(p)$ and hence that $\xi(\tau) = 0$. \square

The first main result in this section is the following.

Theorem 2. *Let M be a Riemannian three-manifold carrying a unit Killing field ξ and p a point in M . Then*

- (1) *M admits a totally geodesic surface passing through p which is everywhere orthogonal to ξ if and only if M has in a neighborhood of p a Riemannian product structure $\Sigma \times I$ of some surface Σ with an interval I and ξ is tangent to the factor I .*
- (2) *M admits a totally geodesic surface passing through p which is everywhere tangent to ξ if and only if there exists a geodesic through p in M on which τ vanishes and which is orthogonal to ξ at p .*
- (3) *The following three assertions are equivalent.*
 - (i) *M admits a totally geodesic surface passing through p which is neither orthogonal nor tangent to ξ at p .*
 - (ii) *There is in a neighborhood of p in M an orthogonal decomposition $\xi = X_1 + X_2$ of ξ , where X_1 and X_2 are Killing fields without zeros that commute.*
 - (iii) *There exist local coordinates (x, y, z) around p in M with $\xi = \partial_y + \partial_z$, such that the metric takes the form*

$$g = dx^2 + \sin^2 \theta(x) dy^2 + \cos^2 \theta(x) dz^2.$$

Remark 4. Statement (1) is valid in all dimensions. In dimension 3 it is equivalent to the vanishing of τ in a neighborhood of p .

Proof. We denote by $(\phi_t)_{t \in I}$ the local flow of ξ .

(1) Suppose M admits a totally geodesic surface Σ , passing through p , which is everywhere orthogonal to ξ . Restricting Σ and the interval I if necessary, the mapping $(x, t) \in \Sigma \times I \mapsto \phi_t(x) \in M$ parametrizes an open subset of M and is easily checked to be an isometry between the Riemannian product manifold $\Sigma \times I$ and that open set.

Conversely it is clear that if M has locally a product structure of some surface Σ and an interval I to which ξ is tangent then the surfaces $\Sigma \times \{t\}$ are totally geodesic and orthogonal to ξ at each point.

(2) Suppose $s \in J \mapsto \gamma(s)$ is a geodesic of M parametrized by arc length such that $\gamma(t_0) = p$ and $\langle \gamma'(t_0), \xi \rangle = 0$ for some $t_0 \in J$ and $\tau(\gamma(s)) = 0$ for all $s \in J$. Note first that γ is everywhere orthogonal to ξ . Indeed, for all $s \in J$

$$\frac{d}{ds} \langle \gamma'(s), \xi \rangle = \left\langle \frac{D\gamma'}{ds}(s), \xi \right\rangle + \langle \gamma'(s), \tau(\gamma(s))(\gamma'(s) \times \xi) \rangle = 0.$$

The mapping $(s, t) \mapsto \phi_t(\gamma(s))$ parametrizes a surface Σ in M and $\tau|_\Sigma = 0$.

We now check that Σ is totally geodesic. A unit normal field to Σ is the field $N(s, t) = (d\phi_t)(\gamma'(s)) \times \xi$. Note that ξ commutes with $(d\phi_t)(\gamma'(s))$, so that

$$\tilde{\nabla}_\xi(d\phi_t)(\gamma'(s)) = \tilde{\nabla}_{(d\phi_t)(\gamma'(s))}\xi = 0,$$

where we used that $\tau|_\Sigma = 0$. Therefore

$$\tilde{\nabla}_\xi N = \tilde{\nabla}_\xi(d\phi_t)(\gamma'(s)) \times \xi + (d\phi_t)(\gamma'(s)) \times \tilde{\nabla}_\xi \xi = 0.$$

Since $s \mapsto \phi_t(\gamma(s))$ is a geodesic in M for each t , we have

$$\tilde{\nabla}_{(d\phi_t)(\gamma'(s))}N = \tilde{\nabla}_{(d\phi_t)(\gamma'(s))}(d\phi_t)(\gamma'(s)) \times \xi + (d\phi_t)(\gamma'(s)) \times \tilde{\nabla}_{(d\phi_t)(\gamma'(s))}\xi = 0.$$

It follows that Σ is totally geodesic.

Conversely, suppose M admits a totally geodesic surface Σ passing through p which is everywhere tangent to ξ . As geodesics on Σ are also geodesics on M , it is enough to check that $\tau|_\Sigma = 0$. This is indeed the case: take an arbitrary point q of Σ and a vector X tangent to Σ and linearly independent of ξ . Since Σ is totally geodesic, the vector $\tilde{\nabla}_X \xi = \tau(q)(X \times \xi)$ has to be tangent to Σ , which is only possible if $\tau(q) = 0$.

(3) First, we prove that (i) implies (iii). Let Σ be totally geodesic in M such that ξ is not tangent to Σ at p . Extend T , N , $JT = N \times T$ and ν to a neighborhood of this p in M using the local flow of ξ . Using equations (6), (5) and (4), we have:

$$\begin{aligned} \tilde{\nabla}_{JT}T &= \tilde{\nabla}_{JT}\xi - (JT)(\nu)N = \tau[(JT \times \xi) - \langle JT \times \xi, N \rangle N] = \tau(JT \times \nu N) = \tau\nu T, \\ \tilde{\nabla}_T JT &= \tilde{\nabla}_T(N \times T) = N \times \tilde{\nabla}_T(\xi - \nu N) = N \times \tau(T \times \xi) = \tau\nu T. \end{aligned}$$

It follows that

$$\begin{aligned} [T, JT] &= \tilde{\nabla}_T JT - \tilde{\nabla}_{JT}T = 0, \\ [T, \nu N] &= [T, \xi] = 0, \\ [JT, \nu N] &= [JT, \xi] = 0. \end{aligned}$$

Hence, we can take local coordinates (x, y, z) on M such that $\partial_x = JT$, $\partial_y = T$ and $\partial_z = \nu N$. With respect to these coordinates, the metric takes the form

$$g = (1 - \nu^2)(dx^2 + dy^2) + \nu^2 dz^2.$$

From (5) one has $\partial_y \nu = T(\nu) = 0$ and $\partial_z \nu = \nu N(\nu) = (\xi - T)(\nu) = 0$. After a change of the x -coordinate, we obtain the form for g given in the theorem.

To see that (iii) implies (ii), it suffices to take $X_1 = \partial_y$ and $X_2 = \partial_z$.

It remains to prove that (ii) implies (i). Let u be a unit vector field perpendicular to X_1 and X_2 . We shall first prove that u commutes with X_1 and X_2 . By using that u is perpendicular to X_1 and that X_1 is Killing, we obtain

$$\begin{aligned} \langle [X_1, u], X_1 \rangle &= \langle \tilde{\nabla}_{X_1} u, X_1 \rangle - \langle \tilde{\nabla}_u X_1, X_1 \rangle \\ &= -\langle u, \tilde{\nabla}_{X_1} X_1 \rangle - \langle \tilde{\nabla}_u X_1, X_1 \rangle \\ &= 0. \end{aligned}$$

Furthermore, by using that u is perpendicular to X_2 and that X_1 is Killing, we find

$$\begin{aligned}\langle [X_1, u], X_2 \rangle &= \langle \tilde{\nabla}_{X_1} u, X_2 \rangle - \langle \tilde{\nabla}_u X_1, X_2 \rangle \\ &= -\langle u, \tilde{\nabla}_{X_1} X_2 \rangle + \langle \tilde{\nabla}_{X_2} X_1, u \rangle \\ &= -\langle [X_1, X_2], u \rangle = 0.\end{aligned}$$

Finally, since $\|u\| = 1$ and X_1 is Killing,

$$\langle [X_1, u], u \rangle = \langle \tilde{\nabla}_{X_1} u, u \rangle - \langle \tilde{\nabla}_u X_1, u \rangle = 0.$$

We conclude that $[X_1, u] = 0$. Analogously, we can prove that $[X_2, u] = 0$.

Now consider an integral surface of the distribution spanned by u and X_1 . Of course, this surface is nowhere tangent or orthogonal to $\xi = X_1 + X_2$. We shall prove that it is totally geodesic. It is sufficient to verify that $\tilde{\nabla}_u u$, $\tilde{\nabla}_u X_1$ and $\tilde{\nabla}_{X_1} X_1$ are all perpendicular to X_2 . Using Koszul's formula and the facts that $[X_1, X_2] = [X_1, u] = [X_2, u] = 0$, $\langle X_1, X_2 \rangle = \langle X_1, u \rangle = \langle X_2, u \rangle = 0$ and $\langle u, u \rangle = 1$, gives immediately that $\langle \tilde{\nabla}_u u, X_2 \rangle = 0$ and $\langle \tilde{\nabla}_u X_1, X_2 \rangle = 0$. Finally, using the facts that X_1 and X_2 are orthogonal and that X_2 is Killing, gives $\langle \tilde{\nabla}_{X_1} X_1, X_2 \rangle = -\langle X_1, \tilde{\nabla}_{X_1} X_2 \rangle = 0$. \square

Remark 5. For later use, we note that in the coordinates where the metric takes the given form in case (iii) we have from the proof of the theorem: $\partial_x = JT/\|T\|$, $\partial_y = T$, $\cos \theta(x) = \langle \xi, N \rangle$ and $\|T\| = \sin \theta(x)$.

We now study further the case (3) in Theorem 2. We are able to determine all the totally geodesic surfaces of M in a neighborhood of p in this case. We will need the following result which can be verified by straightforward computations.

Proposition 4. *The Levi-Civita connection of the metric g defined in local coordinates (x, y, z) by*

$$(8) \quad g = dx^2 + \sin^2 \theta(x) dy^2 + \cos^2 \theta(x) dz^2,$$

is given by

$$\begin{aligned}\tilde{\nabla}_{\partial_x} \partial_x &= 0, & \tilde{\nabla}_{\partial_x} \partial_y &= \cot \theta \theta' \partial_y, & \tilde{\nabla}_{\partial_x} \partial_z &= -\tan \theta \theta' \partial_z, \\ \tilde{\nabla}_{\partial_y} \partial_y &= -\cos \theta \sin \theta \theta' \partial_x, & \tilde{\nabla}_{\partial_y} \partial_z &= 0, & \tilde{\nabla}_{\partial_z} \partial_z &= \cos \theta \sin \theta \theta' \partial_x,\end{aligned}$$

Setting $\xi = \partial_y + \partial_z$, it follows that $\tilde{\nabla}_X \xi = -\theta'(x)(X \times \xi)$ for any tangent vector X , where \times stands for the cross product associated with the orientation given by the chart (x, y, z) . Moreover, the scalar curvature of the manifold is $(\theta')^2 - 4 \cot(2\theta) \theta''$.

Our second main result in this section describes the totally geodesic surfaces in M in case (3) in Theorem 2. It characterizes in particular the flat and spherical metrics. It can be compared to a result of E. Cartan (see [2] p. 233). In the three-dimensional case, Cartan's theorem asserts a three-dimensional Riemannian manifold with a totally geodesic surface passing through any point with any specified plane as tangent plane must be a space form. When the manifold admits a unit Killing field, our result says that the existence of very few totally geodesic surfaces suffices to characterize the space forms of non-negative curvature.

Theorem 3. *Let M be a Riemannian three-manifold carrying a unit Killing field ξ and let $p \in M$. Suppose there is a totally geodesic surface Σ_1 passing through p which is neither orthogonal nor tangent to ξ at p . Then:*

- (1) *There is a second totally geodesic surface passing through p which is orthogonal to Σ_1 .*
- (2) *If there exists a third totally geodesic surface through p which is not tangent to ξ at p , then M has constant non-negative sectional curvature in a neighborhood of p and thus is around p isometric to an open subset of the sphere \mathbb{S}^3 with a metric of constant curvature or to an open set of the Euclidean space \mathbb{R}^3 .*
- (3) *If M does not have constant positive curvature near p , then there exists a totally geodesic surface through p which is tangent to ξ at p if and only if $\tau(p) = 0$.*

Proof. From case (3) in Theorem 2 we can find local coordinates (x, y, z) in a neighborhood W of p where the metric takes the form (8), the point p corresponding to the origin. Restricting Σ_1 if necessary we can assume that Σ_1 is given by the equation $z = 0$.

(1) From the above local expression (8) for the metric we see that the surface Σ_0 , defined by the equation $y = 0$, is totally geodesic and is orthogonal to Σ_1 .

(2) Suppose there is a third totally geodesic surface Σ_2 containing p which is not tangent to ξ at p .

We first treat the case when Σ_2 is not orthogonal to ξ at p . We will show that the function τ is constant in a neighborhood of p . This will conclude the proof as this means that, in the coordinates introduced above, $\theta(x) = \alpha x + \beta$ for some constants α and β . If $\alpha = 0$, the metric g is flat. If $\alpha \neq 0$, then it is straightforward to check that g has constant sectional curvature α^2 .

We still denote by Σ_2 the component of $W \cap \Sigma_2$ containing p . Restricting W and replacing Σ_2 by an open subset of it if necessary, we can assume the intersection $\Sigma_1 \cap \Sigma_2$ is connected. For $i = 1, 2$, denote by N_i a unit normal to Σ_i . As before we introduce the vector field T_i tangent to Σ_i and the real valued function ν_i on Σ_i by the orthogonal decomposition

$$\xi = T_i + \nu_i N_i.$$

We again use the same notations to denote the extensions of N_i, T_i and ν_i to W using the flow of ξ . Note that along $\Sigma_1 \cap \Sigma_2$ the vectors N_1 and N_2 are independent, so, up to restricting W if necessary, we can assume that their extensions are also pointwise independent. In the same way, as Σ_1 and Σ_2 are distinct, T_1 and T_2 are independent along $\Sigma_1 \cap \Sigma_2$ and we can assume their extensions are independent in W .

Suppose that ξ, T_1 and T_2 are linearly independent in an open set $U \subset W$. It follows from Proposition 4 and Remark 5 that τ does not depend on T_1 , that is, $T_1(\tau) = 0$. In the same way $T_2(\tau) = 0$. As $\xi(\tau) = 0$, see Lemma 2, we conclude that $\text{grad } \tau = 0$ in U , where grad denotes the gradient on M .

Suppose now that ξ, T_1 and T_2 are (pointwise) linearly dependent in some connected open set $V \subset W$. Let S denote the surface tangent to the distribution spanned by ξ and T_1 which passes through p . From the expression of the metric (8) obtained in Theorem 2 using the surface Σ_1 , we see that the coordinate x is

the signed distance function to the surface S . As we are assuming ξ , T_1 and T_2 are dependent in V , we conclude that we obtain the same coordinate function x when we use Σ_2 in Theorem 2. The Killing fields T_1 and T_2 are tangent to the integral surfaces of the distribution spanned by ξ and T_1 , that is, the level surfaces of the coordinate function x , and so are Killing fields on each of them. Moreover their norms depend only on x (see Remark 5). For each x , the level surface corresponding to x is flat, that is, locally euclidean. The Killing fields ξ , T_1 and T_2 on such a surface correspond to constant fields under an isometry with an open subset of the Euclidean plane since they have constant norms. Therefore we have in V a relation of the form

$$\xi = \alpha_1(x)T_1 + \alpha_2(x)T_2.$$

We next show that the functions α_1 and α_2 are actually constant.

As ξ , T_1 and T_2 are Killing fields, we have for any vector field Y :

$$\alpha'_1(x)\langle T_1, Y \rangle + \alpha'_2(x)\langle T_2, Y \rangle + Y(\alpha_1)\langle T_1, \partial_x \rangle + Y(\alpha_2)\langle T_2, \partial_x \rangle = 0.$$

Taking successively $Y = T_1$ and $Y = T_2$ we get

$$\langle \alpha'_1(x)T_1 + \alpha'_2(x)T_2, T_1 \rangle = \langle \alpha'_1(x)T_1 + \alpha'_2(x)T_2, T_2 \rangle = 0.$$

Since T_1 and T_2 are independent, we conclude that $\alpha'_1(x) = \alpha'_2(x) = 0$, that is, α_1 and α_2 are constants.

Replacing T_i by $\xi - \nu_i N_i$, for $i = 1, 2$, in the decomposition $\xi = \alpha_1 T_1 + \alpha_2 T_2$ we get

$$(9) \quad \xi = \gamma_1 \nu_1 N_1 + \gamma_2 \nu_2 N_2,$$

where $\gamma_i = \alpha_i / (\alpha_1 + \alpha_2 - 1)$, $i = 1, 2$. We have:

$$\widetilde{\nabla}_{\partial_x} \xi = \gamma_1 \nu'_1(x) N_1 + \gamma_2 \nu'_2(x) N_2.$$

As $\nu_i = \cos \theta_i(x)$, we have $\nu'_i(x) = -\theta'_i(x) \sin \theta_i(x) = \tau \sin \theta_i(x)$, $i = 1, 2$. So

$$\widetilde{\nabla}_{\partial_x} \xi = \tau \{ \gamma_1 \sin \theta_1(x) N_1 + \gamma_2 \sin \theta_2(x) N_2 \}.$$

Taking the inner product of both sides with ξ we obtain:

$$\tau \{ \gamma_1 \sin \theta_1(x) \nu_1 + \gamma_2 \sin \theta_2(x) \nu_2 \} = 0.$$

That is,

$$\tau \{ \gamma_1 \sin 2\theta_1(x) + \gamma_2 \sin 2\theta_2(x) \} = 0.$$

Suppose τ does not vanish in some open set $V_0 \subset V$. Then on V_0 :

$$(10) \quad \gamma_1 \sin 2\theta_1(x) + \gamma_2 \sin 2\theta_2(x) = 0.$$

Taking the derivative we get:

$$(11) \quad \gamma_1 \cos 2\theta_1(x) + \gamma_2 \cos 2\theta_2(x) = 0.$$

By (9), $(\gamma_1, \gamma_2) \neq (0, 0)$, so the determinant of the system in the unknowns γ_1 and γ_2 formed by equations (10) and (11) has to vanish, that is,

$$\sin 2(\theta_2(x) - \theta_1(x)) = 0.$$

However, the quantity $\theta_2(x) - \theta_1(x)$, which is a constant since $\theta'_1(x) = -\tau = \theta'_2(x)$, is neither equal to 0 nor to $\pm\pi/2$. Indeed, otherwise this would imply that Σ_2 locally coincides with Σ_1 or Σ_0 , but this contradicts the assumption that Σ_2 is a totally geodesic surface through p , different from Σ_0 and Σ_1 . Consequently $\tau \equiv 0$ in V .

Summarizing we have shown that $\text{grad } \tau = 0$ on an open dense set in a neighborhood of p in M^3 . Therefore τ is constant near p . This concludes the proof of (2) when Σ_2 is not orthogonal to ξ at p .

Suppose now that Σ_2 is orthogonal to ξ at p . Let $S_0 \subset \Sigma_2$ denote the subset where Σ_2 is not orthogonal to ξ . By the above argument, the function τ is locally constant on S_0 . Suppose now that Σ_2 is orthogonal to ξ in an open set $S_1 \subset \Sigma_2$. So ξ is the unit normal to Σ_2 on S_1 . As Σ_2 is totally geodesic, by the formula in Lemma 2, we have $\tau \equiv 0$ on S_1 . Consequently, denoting by grad^{Σ_2} the gradient on Σ_2 , we have $\text{grad}^{\Sigma_2} \tau = 0$ on an open dense subset of Σ_2 and so τ is constant on Σ_2 . As ξ is transversal to Σ_2 and $\xi(\tau) = 0$, we conclude again that τ is constant in a neighborhood of p in M^3 . This concludes the proof of (2).

(3) Suppose $\tau(p) = 0$, that is, $\theta'(0) = 0$. Then from Proposition 4 the surface given by $x = 0$ is totally geodesic. Conversely, suppose there is a connected totally geodesic surface Σ through p which is tangent to ξ at p . We may assume Σ is contained in the coordinate neighborhood where the metric on M takes the form (8). By the same arguments as in (2), we get that τ is constant on any connected open subset of Σ where ξ is not tangent to Σ . Moreover τ vanishes on any open subset of Σ where ξ is tangent to Σ (see the proof of (2) in Theorem 2). As previously, we conclude that τ is constant on Σ .

Denote by π the projection on the x -axis. We consider three cases:

- *First case:* $I := \pi(\Sigma)$ contains an open interval containing 0. It follows that τ (which depends only on x) is constant in a neighborhood of p . So M is, near p , flat or has constant positive curvature. The second possibility is excluded by hypothesis and consequently τ is identically zero near p .

- *Second case:* $I = \{0\}$, that is, $\Sigma \subset \{x = 0\}$. From the equations in Proposition 4, we see that the surface $\{x = 0\}$ is totally geodesic if and only if $\theta'(0) = 0$, that is, if and only if $\tau(p) = 0$.

- *Third case:* $\pi(\Sigma) \subset [0, +\infty)$ or $\pi(\Sigma) \subset (-\infty, 0]$. This means the surface Σ is on one side of the surface $\{x = 0\}$. The extrinsic curvature of the surface $\{x = 0\}$ is $K_{ext} = -(\theta'(0))^2 = -\tau(p)^2$ as is seen from Proposition 4. It is therefore a saddle surface if $\tau(p) \neq 0$ and we are led in this case to a contradiction since it is a general fact that a totally geodesic surface tangent to a saddle surface at a point cannot lie on one side of it near the tangency point. So necessarily $\tau(p) = 0$. \square

5. PROPERTIES OF THE THREE-SPACES

In this section we shall discuss some properties of three-dimensional spaces M^3 with a unit Killing field ξ that admit totally geodesic surfaces which are neither orthogonal nor tangent to ξ . From Theorem 2 we know that such a manifold locally admits a metric of type

$$(12) \quad g = dx^2 + \sin^2 \theta(x) dy^2 + \cos^2 \theta(x) dz^2,$$

with $\xi = \partial_y + \partial_z$.

The following result, which can be checked through straightforward computations, states that these three-spaces admit Riemannian submersions onto a surface with a Killing field.

Proposition 5. *Given M^3 as above, consider a surface M^2 with local coordinates (u, v) and metric*

$$du^2 + \frac{1}{4} \sin^2(2\theta(u)) dv^2.$$

Then the mapping $\pi : M^3 \rightarrow M^2 : (x, y, z) \mapsto (u, v) = (x, y - z)$ is a Riemannian submersion whose fibers are integral curves of the unit Killing field $\xi = \partial_y + \partial_z$. Remark that the Gaussian curvature of M^2 is $K = 4(\theta')^2 - 2 \cot(2\theta)\theta''$.

Let us now study some global properties of M^3 . In particular we want to investigate which manifolds admit a smooth metric which, in local coordinates, is given by (12).

We will first recall two lemmas from [8] on a class of more general doubly warped products

$$(13) \quad (I \times \mathbb{S}^p \times \mathbb{S}^q, dx^2 + \varphi^2(x)g_{\mathbb{S}^p} + \psi^2(x)g_{\mathbb{S}^q}),$$

where $I \subseteq \mathbb{R}$ is an open interval and $g_{\mathbb{S}^p}$ and $g_{\mathbb{S}^q}$ are the standard Riemannian metrics on \mathbb{S}^p and \mathbb{S}^q .

Lemma 3 ([8]). *If $\varphi : (0, b) \rightarrow (0, \infty)$ is smooth and $\varphi(0) = 0$, then the metric in (13) is smooth at $x = 0$ if and only if $\varphi^{(even)}(0) = 0$, $\varphi'(0) = 1$, $\psi(0) > 0$ and $\psi^{(odd)}(0) = 0$. In this case, the topology near $x = 0$ is $\mathbb{R}^{p+1} \times \mathbb{S}^q$.*

Lemma 4 ([8]). *If $\varphi : (0, b) \rightarrow (0, \infty)$ is smooth and $\varphi(b) = 0$, then the metric in (13) is smooth at $x = b$ if and only if $\varphi^{(even)}(b) = 0$, $\varphi'(b) = -1$, $\psi(b) > 0$ and $\psi^{(odd)}(b) = 0$. In this case, the topology near $x = b$ is also $\mathbb{R}^{p+1} \times \mathbb{S}^q$.*

These results allow us to prove that a smooth metric of type (12) exists on the simply connected manifolds \mathbb{S}^3 , $\mathbb{S}^2 \times \mathbb{R}$ and \mathbb{R}^3 .

Proposition 6. *If $\theta : [0, b] \rightarrow \mathbb{R}$ is a smooth function such that $\theta^{-1}\{0\} = \{0\}$ and $\theta^{-1}\{\pi/2\} = \{b\}$, then the metric (12) defines a smooth metric on \mathbb{S}^3 if and only if $\theta'(0) = \theta'(b) = 1$ and $\theta^{(2k)}(0) = \theta^{(2k)}(b) = 0$ for any positive integer k .*

Proof. It follows from the assumptions on θ that $\theta(0, b) = (0, \pi/2)$. Hence, the functions $\varphi = \sin \theta$ and $\psi = \cos \theta$ are strictly positive on $(0, b)$. Lemma 3 and Lemma 4 yield that (12) then gives rise to a smooth metric on \mathbb{S}^3 if and only if the conditions of Lemma 3 are satisfied at $x = 0$ and the conditions of Lemma 4 are satisfied at $x = b$, with the roles of φ and ψ interchanged.

It is easy to see that the condition $\varphi'(0) = 1$ is equivalent to $\theta'(0) = 1$ and that $\psi(0) > 0$ is automatically satisfied. Similarly $\psi'(b) = -1$ if and only if $\theta'(b) = 1$ and $\varphi(b) > 0$ is automatically satisfied. The remaining conditions are thus

$$\varphi^{(even)}(0) = 0, \quad \psi^{(odd)}(0) = 0, \quad \psi^{(even)}(b) = 0, \quad \varphi^{(odd)}(b) = 0.$$

After a computation and using that $\theta'(0) = \theta'(b) = 1$, one sees that these conditions are equivalent to $\theta^{(2k)}(0) = \theta^{(2k)}(b) = 0$ for any integer $k > 0$. \square

Remark 6. The function $\theta(x) = x$ satisfies the conditions given in Proposition 6. In this case, the metric (12) corresponds to the standard metric on \mathbb{S}^3 and the Riemannian submersion of Proposition 5 is the classical Hopf fibration.

Proposition 7. *If $\theta : [0, b] \rightarrow [0, \infty)$ is a smooth function such that $\theta^{-1}\{0\} = \{0, b\}$ and $\theta^{-1}\{\pi/2\} = \emptyset$, then the metric (12) defines a smooth metric on $\mathbb{S}^2 \times \mathbb{R}$ if and only if $\theta'(0) = -\theta'(b) = 1$ and $\theta^{(2k)}(0) = \theta^{(2k)}(b) = 0$ for any non-negative integer k .*

Proof. Remark that the functions $\varphi = \sin \theta$ and $\psi = \cos \theta$ are positive on $(0, b)$. Hence, (12) defines a smooth metric on $\mathbb{S}^2 \times \mathbb{R}$ if and only if the conditions of Lemma 3 and Lemma 4 are satisfied. We can now proceed in an analogous way as in the proof of Proposition 6 to obtain the result. \square

Proposition 8. *If $\theta : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function such that $\theta^{-1}\{k\pi \mid k \in \mathbb{Z}\} = \theta^{-1}\{\pi/2 + k\pi \mid k \in \mathbb{Z}\} = \emptyset$, then (12) defines a smooth metric on \mathbb{R}^3 , which is moreover complete.*

Proof. It is clear that the metric is smooth under the given assumptions. To prove completeness, we may assume that $\theta(x) \in (0, \pi/2)$ for all $x \in \mathbb{R}$. Now let $\gamma : [0, T) \rightarrow \mathbb{R}^3 : t \mapsto (\gamma_1(t), \gamma_2(t), \gamma_3(t))$ be a curve which diverges to infinity, i.e., such that $\gamma_1(t)^2 + \gamma_2(t)^2 + \gamma_3(t)^2$ tends to infinity if t tends to T . We have to prove that the length of this curve with respect to the metric (12),

$$L(\gamma) = \int_0^T \sqrt{(\gamma'_1(t))^2 + \sin^2(\theta(\gamma_1(t)))(\gamma'_2(t))^2 + \cos^2(\theta(\gamma_1(t)))(\gamma'_3(t))^2} dt,$$

is infinite. Therefore, we consider two cases.

First, assume that γ_1 is unbounded. In this case, we have

$$L(\gamma) \geq \int_0^T |\gamma'_1(t)| dt \geq \lim_{t \rightarrow T} |\gamma_1(t) - \gamma_1(0)| = \infty.$$

Next, assume that γ_1 is bounded. In that case the function $\theta(\gamma_1(t))$ is bounded away from 0 and $\pi/2$ and hence there exists a real constant $c > 0$ such that $\sin(\theta(\gamma_1(s))) \geq c$ and $\cos(\theta(\gamma_1(s))) \geq c$. This implies that

$$L(\gamma) \geq c \int_0^T \sqrt{(\gamma'_2(t))^2 + (\gamma'_3(t))^2} dt = \infty.$$

The last equality is due to the fact that the integral appearing on the left hand side is the Euclidean length of the projection of the curve γ onto the (y, z) -plane. Since γ diverges to infinity but γ_1 is bounded, this projection must have infinite length. \square

It is possible to check, using for instance our Theorem 3, that in the examples of Proposition 6 and Proposition 7, through the points where $x = 0$ or $x = b$, there is no totally geodesic surface which is not tangent to the unit Killing field unless the function θ' is constant in a neighborhood of $x = 0$ and $x = b$, respectively. This is not a mere coincidence. We actually have the following global result.

Theorem 4. *Let M be a connected and simply connected complete Riemannian three-manifold carrying a unit Killing field ξ . Suppose that:*

- (1) *no open subset of M has constant non-negative curvature,*
- (2) *through each point of M there passes a totally geodesic surface which is neither orthogonal nor tangent to ξ .*

Then M is isometric to \mathbb{R}^3 endowed with the metric:

$$ds^2 = dx^2 + \sin^2 \theta(x) dy^2 + \cos^2 \theta(x) dz^2,$$

where $\theta : \mathbb{R} \rightarrow (0, \pi/2)$ is a smooth function whose derivative θ' is not constant on any interval. Moreover $\xi = \partial_y + \partial_z$.

Proof. Let p_0 be a fixed point in M . By Theorems 2 and 3, ξ admits an orthogonal decomposition, $\xi = X_1 + X_2$, in a neighborhood of p_0 , where X_1 and X_2 are two Killing fields which commute and have no zeros. Moreover this decomposition is unique up to ordering of X_1 and X_2 . By the same theorems, given any point $p \in M$ and any continuous path γ joining p_0 to p , we can extend continuously this decomposition along γ till the point p . As M is simply connected, by a standard monodromy argument, the decomposition we obtain at p is independent on the choice of the path γ . We get in this way a global orthogonal decomposition $\xi = X_1 + X_2$, where X_1 and X_2 are now global smooth Killing vector fields on M which commute and have no zeros.

Denote by u a unit vector field on M orthogonal to X_1 and X_2 . Such a global smooth vector field exists since M is orientable. From the proof of (3)-(ii) of Theorem 2, we know that the distribution spanned by u and X_1 is integrable and its integral surfaces are totally geodesic. Therefore the manifold M admits a foliation \mathcal{F} by totally geodesic surfaces. Let \mathcal{F}^\perp be the orthogonal foliation, that is, the foliation by the orbits of X_2 . By a result of Carrière and Ghys [4], $(\mathcal{F}, \mathcal{F}^\perp)$ is a product. This means there is a diffeomorphism between M and the product $\Sigma \times \mathbb{R}$, where Σ is any fixed leaf of \mathcal{F} , sending the leaves of \mathcal{F} to $\Sigma \times \{*\}$ and those of \mathcal{F}^\perp to $\{*\} \times \mathbb{R}$. Denote by z the coordinate on the \mathbb{R} factor. Under this diffeomorphism, the vector field X_2 therefore corresponds to the field $f(z)\partial_z$ for some function f . So, up to reparameterizing the \mathbb{R} factor, we can assume that X_2 corresponds to the field ∂_z . It is clear that Σ is simply connected and is therefore, topologically, either a plane or a sphere. X_1 is vector field on Σ which has no zeros, so Σ is topologically a plane. It is moreover not difficult to check that Σ is complete.

Fix an orientation on Σ and denote by J the rotation over 90 degrees in $T\Sigma$. As in the proof of (3) of Theorem 2, we consider on Σ the fields X_1 and JX_1 which commute, $[X_1, JX_1] = 0$, and are complete since they have bounded norms and Σ is complete. It follows that we can find a global chart for Σ with domain \mathbb{R}^2 and $\partial_x = JX_1$ and $\partial_y = X_1$ for the standard coordinates (x, y) on \mathbb{R}^2 . We include a proof of this fact for completeness. Let $(\phi_x)_{x \in \mathbb{R}}$ and $(\psi_y)_{y \in \mathbb{R}}$ be the flows of JX_1 and X_1 , respectively. Consider the mapping:

$$(x, y) \in \mathbb{R}^2 \rightarrow F(x, y) = (\phi_x \circ \psi_y)(p_0) \in \Sigma.$$

F is a local diffeomorphism with $(dF)(\partial_x) = JX_1$ and $(dF)(\partial_y) = X_1$. We will show it is a global diffeomorphism, this will provide the global chart we want.

- F is one-to-one: by contradiction, suppose (x_1, y_1) and (x_2, y_2) are distinct points in \mathbb{R}^2 with $F(x_1, y_1) = F(x_2, y_2)$. Assume that $x_1 = x_2$ and $y_1 \neq y_2$, then the orbit of X_1 through $\phi_{x_1}(p_0)$ will be closed and will bound a disk in Σ inside which necessarily X_1 will have a zero, which is a contradiction. The case $x_1 \neq x_2$ and $y_1 = y_2$ is similar. Assume now that $x_1 \neq x_2$ and $y_1 \neq y_2$. Set $p_1 = (\phi_{x_1} \circ \psi_{y_1})(p_0) = (\phi_{x_2} \circ \psi_{y_2})(p_0)$. The orbit of JX_1 through p_0 and the orbit of X_1 through p_1 intersect at two distinct points, namely $\phi_{x_1}(p_0)$ and $\phi_{x_2}(p_0)$. However an orbit of JX_1 can intersect an orbit of X_1 at most once. Indeed let γ_1 and γ_2 be orbits of X_1 and JX_1 , respectively. Assume they intersect more than once. Then there will be a bounded disk Ω in Σ with boundary the union of an arc $\alpha_1 \subset \gamma_1$ and an arc $\alpha_2 \subset \gamma_2$ with common endpoints p and q . We assume that along α_2 , the field X_1 points into Ω . The case when X_1 points outside Ω can be treated in a similar way. Consider any point q_1 on α_2 distinct from p and q . The half orbit $\beta := \{\psi_t(q_1), t > 0\}$ of X_1 starting from q_1 will be entirely contained

in Ω . It follows from the Poincaré-Bendixon theorem, see for instance [5], that the accumulation set of β must contain a zero or a closed orbit of X_1 , which is again a contradiction.

-*F is onto*: since F is a local diffeomorphism, the image $F(\mathbb{R}^2)$ is open in Σ , so to conclude it is enough to see that it is closed. Let $(x_n, y_n), n \in \mathbb{N}$, be a sequence of points in \mathbb{R}^2 with $F(x_n, y_n) \rightarrow p_\infty \in \Sigma$ as $n \rightarrow \infty$. For $\epsilon > 0$ small enough, the mapping $(x, y) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \rightarrow (\phi_x \circ \psi_y)(p_\infty) \in \Sigma$ is an embedding with image an open neighborhood V of p_∞ . For n fixed and big enough, we have $F(x_n, y_n) \in V$ and so there is $(x, y) \in (-\epsilon, \epsilon) \times (-\epsilon, \epsilon)$ such that $(\phi_x \circ \psi_y)(p_\infty) = (\phi_{x_n} \circ \psi_{y_n})(p_0)$. Therefore $p_\infty = \phi_{x_n-x} \circ \psi_{y_n-y}(p_0)$ lies in $F(\mathbb{R}^2)$.

In the global coordinates (x, y, z) , like in the proof of (3) in Theorem 2 and with the same notations, the metric on M writes:

$$ds^2 = (1 - \nu(x)^2)(dx^2 + dy^2) + \nu(x)^2 dz^2,$$

where $\nu(x) = \|X_2\|^2$ is a function of x alone. We now make the change of coordinate $\bar{x}(x) = \int \sqrt{1 - \nu(x)^2} dx$. By the completeness of the metric g , the function \bar{x} is a bijection from \mathbb{R} onto \mathbb{R} . Setting $\nu(x) = \cos \theta(\bar{x})$ for some smooth function $\theta : \mathbb{R} \rightarrow (0, \pi/2)$, the metric writes in the global coordinates (\bar{x}, y, z) :

$$ds^2 = d\bar{x}^2 + \sin^2 \theta(\bar{x}) dy^2 + \cos^2 \theta(\bar{x}) dz^2.$$

The condition (1) in the statement means precisely that θ' is not constant on any interval (see the proof of (2) in Theorem 3). \square

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